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Specification of Finite Effect Algebras[†]

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We study and relate five basic methods for specifying or describing a finite effect algebra, indicate some computational algorithms for dealing with effect algebras so specified, and mention in passing some open questions that await solution.

1. INTRODUCTION

The contemporary quantum theory of measurement employs measures that take values in the standard effect algebra of self-adjoint Hilbert space operators between zero and the identity [1]. The infinite-dimensional standard effect algebras in which these positive-operator-valued (POV) measures assume their values harber many structural mysteries. Small finite effect algebras are amenable to exhaustive study with the aid of computer software, and the results of these studies can cast considerable light on the structure of the standard effect algebra and thus on the theory of POV-measures.

The evolving discipline of *computational quantum logic* is devoted to the development of efficient algorithms for dealing with finite effect algebras. However, before an effect algebra L can be processed by computer software, it is necessary to give an explicit specification or description of L. Furthermore, one of the tasks of computational quantum logic is to develop efficient algorithms for covering one description of L (e.g., in terms of multiplicity vectors) to a more perspicuous description (e.g., as an interval in a partially ordered Abelian group).

[†]This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

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In this largely expository paper we focus on the question of how finite effect algebras can be specified or described. We assume that the reader is familiar with the definition and the basic theory of effect algebras [2–8]. To keep the length of the paper within bounds, we omit all proofs. Omitted proofs are either straightforward or can be found in the cited literature.

There are perhaps five basic (not necessarily mutually exclusive) methods for specifying a particular effect algebra L: (1) *intrinsically*, (2) *by an edge-labeled Hasse diagram*, (3) *by multiplicity vectors*, (4) *as subset of an Abelian group*, and (5) *as an interval in a partially ordered Abelian group*. We discuss methods 3, 4, and 5, respectively, in Sections 2, 3, and 5 below. In Section 4 we review the notion of the universal group for L. In the course of our discussion, we mention a number of intriguing open questions. Here we briefly consider methods 1 and 2.

By an *intrinsic* specification of L (method 1 above), we mean a description of L within or in terms of a conventional mathematical structure such as an operator algebra, a measure space, a group, or a combinatorial design. For example, a standard effect algebra is described as a certain system of operators on a Hilbert space. Another example would be the set of all subsets of the unit interval [0, 1] that have rational Lebesgue measure. Note that neither of these descriptions is particularly amenable to processing by computer software.

Although an effect algebra L is a partially ordered set (poset), its algebraic structure is not determined by its poset structure. Even the four-element Boolean algebra, regarded as a poset, can be organized into an effect algebra in two different ways. Thus, the usual Hasse diagram does not suffice to specify the algebraic structure of L.

A complete description of a finite effect algebra L can be secured by labeling the *edges* of its Hasse diagram as follows (method 2 above): A rising line segment from vertex p to vertex q in the Hasse diagram indicates that $p, q \in L$ and that q covers p. The fact that q covers p means that there is a uniquely determined atom $a \in L$ such that $p \perp a$ and $p \oplus a = q$. By labeling the line segment from p up to q by the atom a, this information is incorporated into the diagram. By so labeling all of the line segments (edges), complete information about the algebraic structure of L is encoded into the *edge-labeled Hasse diagram*. In studying a finite effect algebra L, one is rarely in possession at the outset of the complete edge-labeled Hasse diagram. Rather, the information in the diagram is usually part of the desired *output* of a computer algorithm.

In what follows, we turn our attention to the remaining methods (3–5) for specifying or describing a finite effect algebra. We denote the system of integers by $\mathbf{Z}$ and we define $\mathbf{Z}^+ := \{z \in \mathbf{Z} | 0 \leq z\}$. (The symbol := means equals by definition.) If n is a positive integer, we understand that

$$\mathbf{Z}^n := \{(z_1, z_2, \ldots, z_n) | z_i \in \mathbf{Z} \text{ for } i = 1, 2, \ldots, n\}$$

is organized into a partially ordered Abelian group under coordinatewise addition with

$$(\mathbf{Z}^+)^n := \{(z_1, z_2, \ldots, z_n) | z_i \in \mathbf{Z}^+ \text{ for } i = 1, 2, \ldots, n\}$$

as the positive cone. Vectors in $\mathbf{Z}^n$ are denoted by lowercase boldface letters and *the standard (Kronecker) basis vectors* $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n \in (\mathbf{Z})^n$ are defined by

$$\mathbf{d}_i := (0, 0, \ldots, 0, 1, 0, \ldots, 0)$$

with the 1 in the ith coordinate for $i = 1, 2, \ldots, n$.

For the remainder of the paper, *we assume that L is a finite effect algebra with zero* 0, *unit* $u \neq 0$, *and orthosupplementation* $p \mapsto p'$. Also, *the atoms in L are denoted by* $a_1, a_2, \ldots, a_n$.

2. MULTIPLICITY VECTORS

Every element $p \in L$ can be written as an orthocombination $p = \bigoplus_{i=1}^n p_i a_i$ of the atoms $a_1, a_2, \ldots, a_n$ with coefficients $p_1, p_2, \ldots, p_n \in \mathbf{Z}^+$. In particular, the unit u can be so written.

Definition 2.1. A *multiplicity vector for L* with respect to the atoms $a_1, a_2, \ldots, a_n$ is a vector $\mathbf{t} = (t_1, t_2, \ldots, t_n) \in (\mathbf{Z}^+)^n$ such that $u = \bigoplus_{i=1}^n t_i a_i$. The set consisting of all such multiplicity vectors is denoted by $T \subseteq (\mathbf{Z}^+)^n$ and is called the *total set of multiplicity vectors for L* with respect to $a_1, a_2, \ldots, a_n$.

The set T is necessarily finite, and the entire structure of the effect algebra L, up to an isomorphism, is encoded in T. Thus, *L is specified by the finite set T of vectors over* $(\mathbf{Z}^+)^n$. The structure of L can be extracted from T by proceeding as follows.

Definition 2.2. Let $T\!\downarrow := \{\mathbf{p} \in (\mathbf{Z}^+)^n | \mathbf{p} \leq \mathbf{t} \text{ for some } \mathbf{t} \in T\}$.

For $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in T\!\downarrow$, define $\sigma(\mathbf{p}) \in L$ by

$$\sigma(\mathbf{p}) := \bigoplus_{i=1}^n p_i a_i$$

Let $\mathscr{L} := \{\sigma^{-1}(p) | p \in L\}$. The elements of $\mathscr{L}$ are called *perspectivity classes* in $T\!\downarrow$.

Define $\iota: L \to \mathscr{L}$ by $\iota(p) := \sigma^{-1}(p)$ for all $p \in L$.

Evidently the mapping $\sigma: T\!\downarrow \to L$ is a surjection, $\mathscr{L}$ is a partition of $T\!\downarrow$ into equivalence classes, and $\iota: L \to \mathscr{L}$ is a bijection. Using the bijection ι, *we can and do organize $\mathscr{L}$ into an effect algebra* $(\mathscr{L}, 0, T, \oplus)$ *with zero*

$\mathbf{0} = \sigma^{-1}(0)$ *and unit* $T = \sigma^{-1}(u)$ *in such a way that* $\iota: L \to \mathscr{L}$ *is an effect-algebra isomorphism*.

According to the following theorem, the structure of the effect algebra $\mathscr{L}$, hence also the structure of L, can be assessed using only the information implicit in the set T.

Theorem 2.3. If $\mathbf{p}, \mathbf{q} \in T\!\downarrow$, then $\mathbf{p}$ and $\mathbf{q}$ belong to the same perspectivity class in $\mathscr{L}$ iff there is a vector $\mathbf{r} \in T\!\downarrow$ such that $\mathbf{p} + \mathbf{r} \in T$ and $\mathbf{q} + \mathbf{r} \in T$. If $P, Q \in \mathscr{L}$, then $P \perp Q$ iff there exist $\mathbf{p} \in P$, $\mathbf{q} \in Q$ with $\mathbf{p} + \mathbf{q} \in T\!\downarrow$, in which case $\mathbf{p} + \mathbf{q} \in T\!\downarrow$ holds for all $\mathbf{p} \in P$ and all $\mathbf{q} \in Q$. If $P, Q \in \mathscr{L}$ with $P \perp Q$, then

$$P \oplus Q = \{\mathbf{p} + \mathbf{q} \,|\, \mathbf{p} \in P, \mathbf{q} \in Q\}$$

If $\mathbf{p} \in P \in \mathscr{L}$, then the orthosupplement of P in $\mathscr{L}$ is

$$P' = \{\mathbf{t} - \mathbf{p} \,|\, \mathbf{p} \le \mathbf{t} \text{ and } \mathbf{t} \in T\}$$

The atoms in $\mathscr{L}$, given by $A_i := \iota(a_i)$ for $i = 1, 2, \ldots, n$, are determined by the fact that $d_i \in A_i$. In fact, $A_i = \{d_i\}$.

Theorem 2.3 can be used as the basis for an algorithm for finding the structure, up to an isomorphism, of a finite effect algebra L in terms of its total set T of multiplicity vectors. See Section 6 of ref. 4 for the details.

Example 2.4. The eight-element effect algebra

$$F_8 = \{0, u, a, b, c, a', b', c'\}$$

has three atoms a, b, c and three coatoms a', b', c' and the total set of multiplicity vectors for F_8 is

$$T = \{(2, 0, 1), (1, 2, 0)\}$$

In other words, there are two and only two ways to write the unit u as an orthocombination of the atoms, namely

$$2a \oplus c = u \qquad \text{and} \qquad a \oplus 2b = u$$

Evidently, $T\!\downarrow$ consists of the 10 vectors

$$(0, 0, 0), \quad (1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1), \quad (1, 0, 1)$$
$$(1, 1, 0), \quad (0, 2, 0), \quad (2, 0, 0), \quad (2, 0, 1), \quad (1, 2, 0)$$

There are eight perspectivity classes in $T\!\downarrow$, and they correspond to the elements of F_8 as follows:

$$\iota(0) = \{(0, 0, 0)\}, \qquad \iota(u) = T = \{(2, 0, 1), (1, 2, 0)\}$$
$$\iota(a) = \{(1, 0, 0)\}, \qquad \iota(b) = \{(0, 1, 0)\},$$

$$\iota(c) = \{(0, 0, 1)\}, \qquad \iota(a') = \{(1, 0, 1), (0, 2, 0)\},$$
$$\iota(b') = \{(1, 1, 0)\}, \qquad \iota(c') = \{(2, 0, 0)\}$$

Orthogonalities and orthosums can be deduced from this information. For instance, $(1, 0, 0) \in \iota(a)$, $(0, 1, 0) \in \iota(b)$, and $(1, 0, 0) + (0, 1, 0) = (1, 1, 0) \in \iota(b')$, so $a \perp b$ with $a \oplus b = b'$. Also, $(0, 1, 0) \in \iota(b)$, $(0, 0, 1) \in \iota(c)$, and $(0, 1, 0) + (0, 0, 1) = (0, 1, 1) \notin T\downarrow$, whence $b \not\perp c$, and so on. Proceeding in this way, one can extract the necessary information to construct the edge-labeled Hasse diagram for F_8. ■

In the sequel, we use F_8 as a running example to illustrate various methods for specifying a finite effect algebra. The Hasse diagram and the edge-labeled Hasse diagram for F_8 are given in Fig. 1.

3. EFFECT GROUPS AND GROUP REALIZATIONS

In what follows, all Abelian groups will be written additively. If H is an Abelian group and $E \subseteq H$, we write $\langle E \rangle$ for the subgroup of H generated by E.

Definition 3.1. An *effect group* is a triple (H, E, v) consisting of an Abelian group H, a subset $E \subseteq H$ such that $\langle E \rangle = H$, and an element $v \in E$ such that for all $a, b, c \in E$:

(i) $a + b, a + b + c \in E \Rightarrow b + c \in E$.
(ii) $v - a \in E$.
(iii) $v + a \in E \Rightarrow a = 0$.

Theorem 3.2. Let (H, E, v) be an effect group. Then E can be organized into an effect algebra $(E, 0, v, \oplus)$ by defining $a \oplus b$ for $a, b \in E$ iff $a + b \in E$, in which case $a \oplus b := a + b$.

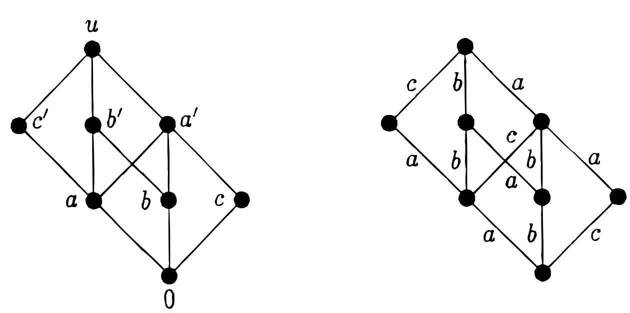

Fig. 1. The usual Hasse diagram for F_8 and the edge-labeled Hasse diagram for F_8.

The condition in Definition 3.1 that $\langle E \rangle = H$ just ensures that there are no elements in H that are "algebraically irrelevant" to the structure of the effect algebra E. If all the other conditions in Definition 3.1 hold, then this condition can be enforced simply by replacing H by $\langle E \rangle$.

Definition 3.3. An effect algebra L is said to have a *group realization* (H, E, v) iff (H, E, v) is an effect group and L is isomorphic to the effect algebra E.

If L has a group realization, then (up to an isomorphism) *L can be specified as a subset of an Abelian group H and the algebraic structure of L is determined by the structure of H*. The following example shows that the effect algebra F_8 in Example 2.4 has a group realization. The source of this particular realization is addressed in Example 3.6 below.

Example 3.4. Let E be the set consisting of the eight vectors

$$(0, 0), \quad (2, 0), \quad (1, 1), \quad (0, 2), \quad (4, 0), \quad (3, 1), \quad (2, 2), \quad (4, 2)$$

in the group $\mathbf{Z}^2$ and let $\mathbf{v} := (4, 2)$. Then $(\mathbf{Z}^2, E, \mathbf{v})$ is an effect group. Organize E into an effect algebra as in Theorem 3.2. Then the effect algebra F_8 is isomorphic to E under the mapping $0 \mapsto (0, 0)$, $a \mapsto (2, 0)$, $b \mapsto (1, 1)$, $c \mapsto (0, 2)$, $a' \mapsto (2, 2)$, $b' \mapsto (3, 1)$, $c' \mapsto (4, 0)$, and $u \mapsto \mathbf{v} = (4, 2)$.

If $[0, 1]$ denotes the closed unit interval in the additive Abelian group $\mathbf{R}$ of real numbers, then $(\mathbf{R}, [0, 1], 1)$ is an effect group, and $[0, 1]$ is thus organized into an effect algebra called the *standard scale algebra*. A morphism $\omega\colon L \mapsto [0, 1]$ is the same thing as a *probability measure* (or *state*) on L. Since L is finite, the set $\Omega(L)$ of all probability measures on L forms a convex polytope. See Section 8 of ref. 6 for a sketch of an algorithm for calculating the set $\partial_e\Omega(L)$ of all extreme points (*pure states*) of $\Omega(L)$ from the set T of multiplicity vectors for L. If $\omega \in \partial_e\Omega(L)$, then $\omega(p)$ is a rational number for all $p \in L$.

Theorem 3.5. Suppose $\Omega(L)$ is a full (i.e., order-determining) set of probability measures on L, let $\partial_e\Omega(L) = \{\omega_1, \omega_2, \ldots, \omega_k\}$, and let $a_1, a_2, \ldots, a_n$ be atoms in L. For each $i = 1, 2, \ldots, k$, write the rational numbers $\omega_i(a_1), \omega_i(a_2), \ldots, \omega_i(a_n)$ in reduced form, let v_i be the least common multiple of the resulting denominators, and define $\epsilon_i\colon L \mapsto \mathbf{Z}^+$ by $\epsilon_i(p) := v_i\omega_i(p)$ for all $p \in L$. Let $H := \mathbf{Z}^k$, define $\eta\colon L \mapsto H$ by $\eta(p) := (\epsilon_1(p), \epsilon_2(p), \ldots, \epsilon^k(p))$ for all $p \in L$, let $E := \eta(L)$, and let $\mathbf{v} := (v_1, v_2, \ldots, v_k)$. Then $(H, E, \mathbf{v})$ is an effect group, η is an H-valued measure on L, and $\eta\colon L \to E$ is an effect-algebra isomorphism.

As a consequence of Theorem 3.5, if L admits a full set of probability measures, then L has a group realization.

Example 3.6. A computer calculation implementing the algorithm given in ref. 6 shows that $\partial_e(F_8) = \{\omega_1, \omega_2\}$ with

$$\omega_1(a) = 1/2, \qquad \omega_1(b) = 1/4, \qquad \omega_1(c) = 0$$

$$\omega_2(a) = 0, \qquad \omega_2(b) = 1/2, \qquad \omega_2(c) = 1$$

$\Omega(F_8)$ is full, and in Theorem 3.5, $k = 2$, $v_1 = 4$, $v_2 = 2$, $H = \mathbf{Z}^2$,

$$\eta(a) = (2, 0), \qquad \eta(b) = (1, 1), \qquad \eta(c) = (0, 2), \qquad \mathbf{v} = (4, 2)$$

Indeed, the effect group $(H, E, \mathbf{v})$ is the effect group in Example 3.4 and the isomorphism $\eta\colon F_8 \to E$ is the isomorphism in that example. ■

4. THE UNIVERSAL GROUP

In general, group realizations are not unique. However, if L admits a group realization, it admits a special group realization $(G, \gamma(L), \gamma(u))$ from which all the others can be derived in the sense that, if (H, E, v) is a group realization of L, there is a uniquely determined group epimorphism $\nu\colon G \to H$ such that $\nu \circ \gamma\colon L \to E$ is an effect-algebra isomorphism. In this section we review the construction of the group G and the G-valued measure $\gamma\colon L \to G$. We maintain the standing notation of Section 2 and we *choose and fix a multiplicity vector* $\mathbf{s} \in T$.

Definition 4.1. Let $\langle T - \mathbf{s}\rangle$ be the subgroup of $\mathbf{Z}^n$ generated by all vectors of the form $\mathbf{t} - \mathbf{s}$ for $\mathbf{t} \in T$, let G be an Abelian group, and let $\xi\colon \mathbf{Z}^n \to G$ be a surjective group homomorphism with kernel $\langle T - \mathbf{s}\rangle$.

It is clear that $\ker(\xi) = \langle T - \mathbf{s}\rangle$ is independent of the choice of the vector $\mathbf{s} \in T$ and G is isomorphic to the quotient group $\mathbf{Z}^n/\langle T - \mathbf{s}\rangle$. If desired, one can take $G = \mathbf{Z}^n/\langle T - \mathbf{s}\rangle$ and let $\xi\colon \mathbf{Z}^n \to \mathbf{Z}^n/\langle T - \mathbf{s}\rangle$ be the natural epimorphism. In any case, we shall refer to $\xi\colon \mathbf{Z}^n \to G$ as the *canonical epimorphism*.

Theorem 4.2. There is a unique mapping $\gamma\colon L \to G$ such that, for all $p \in L$ and all $\mathbf{p} \in \iota(p)$, $\gamma(p) = \xi(\mathbf{p})$. Further, γ has the following properties:

(i) $\gamma \cdot L \to G$ is a group-valued measure on the effect algebra L.
(ii) $\gamma(L)$ generates the group G.
(iii) If K is any Abelian group and $\phi\colon L \to K$ is a K-valued measure on L, there is a unique group homomorphism $\phi^*\colon G \to K$ such that $\phi = \phi^* \circ \gamma$.

Because of property (iii) in Theorem 4.2, the pair (G, γ) is called a *universal group* for the effect algebra L. Since (G, γ) is uniquely determined by L up to an isomorphism, we often allow ourselves to refer to it as *the* universal group for L. Weber [11] and Navara [9] have given examples of finite orthomodular lattices with universal group $G = \{0\}$.

The group G and the G-valued measure γ can be found by standard algorithms for finitely generated Abelian groups and G can be realized as a Cartesian product $G = \mathbf{Z}^r \times J$ of a free Abelian group $\mathbf{Z}^r$ of finite rank r and a finite group J that is a Cartesian product of cyclic groups. The factor J is called the *torsion subgroup* of G and G is said to be *torsion-free* iff $J = \{0\}$.

Definition 4.3. If (G, γ) is the universal group for the effect algebra L and G is isomorphic to $\mathbf{Z}^r \times J$, where J is a finite group, then r is called the *rank* of L. We say that L is *torsion-free* iff $J = \{0\}$.

There are many open questions centered around the question of which effect algebras are torsion-free and which are not. In general, it appears that the torsion-free effect algebras are considerably "better behaved" than the effect algebras with torsion. We do not know an example of an effect algebra with full set of probability measures that is not torsion-free.

Example 4.4. For F_8 (Example 2.4) the universal group is $(\mathbf{Z}^2, \gamma)$ with

$$\gamma(0) = (0, 0), \qquad \gamma(u) = (1, 1)$$
$$\gamma(a) = (1, -1), \qquad \gamma(b) = (0, 1), \qquad \gamma(c) = (-1, 3)$$
$$\gamma(a') = (0, 2), \qquad \gamma(b') = (1, 0), \qquad \gamma(c') = (2, -2)$$

Thus, F_8 is torsion-free with rank 2.

In Example 4.4, note that $\gamma: F_8 \to G$ is an injection. More generally, if (G, γ) is a universal group for L, then γ is an injection iff there are sufficiently many group-valued measures on L to separate the elements of L.

If (G, γ) is a universal group for L and $\lambda: G \to G$ is an automorphism of G, then $(G, \lambda \circ \gamma)$ is again a universal group of L. An automorphism λ of the free Abelian group $\mathbf{Z}^r$ of rank r determines and is determined by a unique $r \times r$ unimodular matrix U, so that, for $z \in \mathbf{Z}^r$, $\lambda(z) = zU$.

Example 4.5. In Example 4.4, let $U = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. Then U is a unimodular matrix that determines an automorphism $z \mapsto \lambda(z) := zU$ of $\mathbf{Z}^2$, so $(\mathbf{Z}^2, \Gamma)$ with $\Gamma := \lambda \circ \gamma$ provides an alternative representation for the universal group of F_8. We have

$$\Gamma(0) = (0, 0), \qquad \Gamma(u) = (3, 2)$$
$$\Gamma(a) = (1, 0), \qquad \Gamma(b) = (1, 1), \qquad \Gamma(c) = (1, 2)$$
$$\Gamma(a') = (2, 2), \qquad \Gamma(b') = (2, 1), \qquad \Gamma(c') = (2, 0)$$

In Example 4.5, Γ maps L into the *standard positive cone* $(\mathbf{Z}^+)^2$ in $\mathbf{Z}^2$.

We have not been able to find an example of a finite torsion-free effect algebra L for which this cannot be done.

By the next theorem, L has a group realization if and only if it has a group realization within its own universal group.

Theorem 4.6. Let (G, γ) be the universal group for L. Then L has a group realization iff $(G, \gamma(L), \gamma(u))$ is an effect group and $\gamma: L \to \gamma(L)$ is an effect-algebra isomorphism.

Example 4.7. Applying Theorem 4.6 to Example 4.5, we let F be the set consisting of the eight vectors

$$(0, 0), \quad (1, 0), \quad (1, 1), \quad (1, 2), \quad (2, 2), \quad (2, 1), \quad (2, 0), \quad (3,2)$$

in $\mathbf{Z}^2$ and we let $\mathbf{w} := (3, 2)$. Thus $(\mathbf{Z}^2, F, \mathbf{w})$ is an effect group and, organizing F into an effect algebra as in Theorem 3.2, we find that the effect algebra F_8 is isomorphic to F under the mapping Γ of Example 4.5.

Note that the group realization $(\mathbf{Z}^2, F, \mathbf{w})$ of F_8 in Example 4.7 and the group realization $(\mathbf{Z}^2, E, \mathbf{v})$ of F_8 in Example 3.4 are *not isomorphic*. Specifically, there is a group homomorphism $\nu: \mathbf{Z}^2 \to \mathbf{Z}^2$ given by $\nu(x, y) := (2x - y, y)$ for $(x, y) \in \mathbf{Z}^2$ such that the restriction of ν to F is an effect algebra isomorphism of F into E; however, $\nu: \mathbf{Z}^2 \to \mathbf{Z}^2$ is not a group isomorphism (it fails to be surjective). Indeed, there is no group isomorphism of $\mathbf{Z}^2$ onto $\mathbf{Z}^2$ that carries F onto E.

Suppose the finite effect algebra L has a group realization. By Theorem 4.6, we can identify L with the subset $\gamma(L) \subseteq G$, where G is a finitely generated Abelian group. Thus *L can be specified as a subset of its universal group G, (G, L, u) is an effect group, and the structure of L can be assessed as in Theorem 3.2.*

5. INTERVAL EFFECT ALGEBRAS

If H is a partially ordered abelian group with positive cone H^+, and if $v \in H^+$, then *the interval* $H^+[0, v] := \{h \in H | 0 \leq h \leq v\}$ can be organized into an effect algebra with unit v as follows: For $h, k \in H^+[0, v]$, $h \oplus k$ is defined iff $h + k \leq v$, in which case $h \oplus k := h + k$. We refer to an effect algebra of the form $H^+[0, v]$, or isomorphic to one of this form, as an *interval effect algebra*.

If L is an interval effect algebra, we can specify L, up to an isomorphism, as an interval in a partially ordered Abelian group. Note that every interval effect algebra admits a group realization, so this method of specification is a special case of the method presented in the first part of Section 3.

Example 5.1. Let $H = \mathbf{Z}^2$ and let H^+ be the set of all vectors $(x, y) \in H$ such that $x, y \in \mathbf{Z}^+$ and $x - y$ is an even integer. Then H is a partially ordered Abelian group with positive cone H^+. If we let $\mathbf{v} = (4, 2) \in H^+$, we find that $H^+ [0, \mathbf{v}]$ consists precisely of the eight vectors

$$(0, 0), \quad (2, 0), \quad (1, 1), \quad (0, 2), \quad (4, 0), \quad (3, 1), \quad (2, 2), \quad (4, 2)$$

Thus $H^+ [0, \mathbf{v}]$ coincides with E in Example 3.4, so F_8 is isomorphic to the interval effect algebra $H^+[0, \mathbf{v}]$.

If H is an Abelian group and $0 \in M \subseteq H$, we define ssg(M) to be the subsemigroup of H generated by M. Thus ssg(M) consists of 0 and all finite sums of the form $m_1 + m_2 + \ldots + m_k$ such that $m_1, m_2, \ldots, m_k \in M$. In Example 5.1, note that $H^+ = \text{ssg}(E)$.

Theorem 5.2. If L is an interval effect algebra and (G, γ) is the universal group of L, then G can be organized into a partially ordered Abelian group with positive cone $G^+ := \text{ssg}(\gamma(L))$. Furthermore, $\gamma\colon L \to G^+[0, \gamma(u)]$ is an effect algebras isomorphism.

In Theorem 5.2, we can use the isomorphism γ to identify L with the interval $G^+[0, \gamma(u)]$. Thus, every interval effect algebra L can be realized as an interval in its own universal group G, partially ordered by $G^+ := \text{ssg}(L)$. Consequently, we have the following procedure.

Procedure 5.3. To check whether L is an interval effect algebra:

1. Use the total set T of multiplicity vectors to calculate the universal group (G, γ).
2. Verify that $(G, \gamma(L), \gamma(u))$ is an effect group.
3. Verify that $\gamma\colon L \to \gamma(L)$ is an effect-algebra isomorphism.
4. Calculate $G^+ := \text{ssg}(\gamma(L))$ in G.
5. Verify that $G^+ \cap (-G^+) \subseteq [0]$.
6. Verify that $G^+[0, \gamma(u)] \subseteq \gamma(L)$.

If the procedure is carried out and all verifications can be made, then L is an interval effect algebra; otherwise, it is not.

The execution of Procedure 5.3 is a bit simpler if L is known to admit a group realization (e.g., by Theorem 3.5, if L admits a full set of probability measures). Indeed, if L admits a group realization, then one can begin by identifying L with $\gamma(L) \subseteq G$ as we do in the next example.

Example 5.4. By Example 5.1 we already know that F_8 is an interval effect algebra. Nonetheless, we carry out Procedure 5.3 for F_8 to illustrate how it works. Using the results of Example 4.5, we identify the effect algebra F_8 with the eight vectors

$$\mathbf{0} = (0, 0), \quad \mathbf{a} = (1, 0), \quad \mathbf{b} = (1, 1), \quad \mathbf{c} = (1, 2)$$
$$\mathbf{u} = (3, 2), \quad \mathbf{a}' = (2, 2), \quad \mathbf{b}' = (2, 1), \quad \mathbf{c}' = (2, 0)$$

in its universal group $G = \mathbf{Z}^2$. To calculate $G^+ = \text{ssg}(F_8)$, it will suffice to calculate

$$G^+ = \text{ssg}(\{\mathbf{a}, \mathbf{b}, \mathbf{c},\}) = \{p\mathbf{a} + q\mathbf{b} + r\mathbf{c} | p, q, r \in \mathbf{Z}^+\}$$
$$= \{(p + q + r, q + 2r) | p, q, r \in \mathbf{Z}^+\}$$

Thus, $(x, y) \in G^+$ iff $\exists p, q, r \in \mathbf{Z}^+$ with $p + q + r = x$ and $q + 2r = y$. Solving the last two equations for p and q in terms of x, y, and r, we find that $p = x - y + r$ and $q = y - 2r$. Thus, we have to solve the system of linear inequalities $x - y + r \geq 0$ and $y - 2r \geq 0$ with the understanding that x, y, and r are nonnegative integers. The solution is $0 \leq y \leq 2x$ with $\max(0, y - x) \leq r \leq y/2$. For instance, a solution is obtained by taking $r = \max(0, y - x)$, provided that $0 \leq y \leq 2x$. Thus, we have

$$G^+ = \{(x, y) \in \mathbf{Z}^2 | 0 \leq y \leq 2x\}$$

Evidently, $G^+ \cap (-G^+) \subseteq \{0\}$, so $G = \mathbf{Z}^2$ is partially ordered by the cone G^+. Further calculation reveals that both (x, y) and $\mathbf{u} - (x, y) = (3 - x, 2 - y)$ belong to G^+ iff (x, y) is one of the eight vectors in F_8.

In carrying out Procedure 5.3, the problem of representing G^+ in some perspicuous way is often computationally challenging. As in Example 5.4, it reduces to solving a system of linear inequalities over $\mathbf{Z}^+$. The Fourier–Motzkin algorithm [10] is an efficient method for solving a system of linear inequalities over the rationals, and in some cases it can be adapted to a solution over $\mathbf{Z}^+$, but the authors are unaware of any general algorithm that will accomplish this.

In Example 5.4, ssg(L) is described by linear inequalities $0 \leq x$ and $x \leq 2y$. The authors do not know of any reasonable necessary or sufficient conditions for this to be so. Note that H^+ in Example 5.1 cannot be described in terms of linear inequalities alone.

REFERENCES

1. Busch, P., Lahti, P., and Mittelstaedt, P., *The Quantum Theory of Measurement*, 2nd ed., Springer-Verlag, Berlin, 1996.
2. Bennett, M. K., and Foulis, D. J., Interval and scale effect algebras, *Advances in Applied Mathematics* **19** (1997) 200–215.
3. Foulis, D. J., and Bennett, M. K., Effect algebras and unsharp quantum logics, *Foundations of Physics* **24** (1994) 1325–1346.
4. Foulis, D. J., Bennett, M. K., and Greechie, R. J., Test groups and effect algebras, *International JournaI of Theoretical Physics* **35** (1996) 1117–1140.

5. Foulis, D. J., Greechie, R. J., and Bennett, M. K., Sums and products of interval algebras, *International Journal of Theoretical Physics* **33** (1994) 2119–2136.
6. Foulis, D. J., and Greechie, R. J., Probability weights and measures on finite effect algebras, *International Journal of Theoretical Physics*, to appear.
7. Greechie, R. J., and Foulis, D. J., The transition to effect algebras, *International Journal of Theoretical Physics* **34** (1995) 1369–1382.
8. Greechie, R. J., Foulis, D. J., and Pulmannová, S., The center of an effect algebra, *Order* **12** (1995) 91–106.
9. Navara, M., An orthomodular lattice admitting no group-valued measures, *Proceedings of the American Mathematical Society* **122** (1994) 7–12.
10. Stoer, J., and Witzgall, C., *Convexity and Optimization in Finite Dimensions I*, Springer-Verlag, New York, 1970.
11. Weber, H., There are orthomodular lattices without nontrivial group-valued measures: A computer based construction, *Journal of Mathematical Analysis and Applications* **183** (1994) 89–93.